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# SOME RESULTS AND PROBLEMS ON ANR'S FOR STRATIFIABLE SPACES

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Stratifiable spaces are also called  $M_3$ -spaces, which were introduced by Ceder [Ce] and renamed by Borges [Bo]. The class  $\mathcal{S}$  of stratifiable spaces contains both metrizable spaces and CW-complexes and has many desirable properties (cf. [Bo]). And CW-complexes are ANR for the class  $\mathcal{S}$  [Ca<sub>1</sub>]. Hence it has been expected that ANR theory for the class  $\mathcal{S}$  is established so successfully as the class  $\mathcal{M}$  of metrizable spaces. An absolute (neighborhood) retract for a class  $\mathcal{C}$  is simply called an AR( $\mathcal{C}$ ) (resp. ANR( $\mathcal{C}$ )). Although ANR( $\mathcal{S}$ )'s have been studied by Borges, Cauty and Miwa, etc., many problems are still left. In this note, we present the result of [GS] and some related problems.

The *join* of spaces  $X$  and  $Y$  is defined as the space

$$X * Y = X \cup X \times Y \times (0, 1) \cup Y$$

admitting the topology generated by all open sets in the product space  $X \times Y \times (0, 1)$  and the following sets:

$$U \cup U \times Y \times (0, t) \quad \text{and} \quad X \times V \times (t, 1) \cup V,$$

where  $U$  and  $V$  are open in  $X$  and  $Y$ , respectively, and  $0 < t < 1$ . In [Ca<sub>3</sub>], this join is denoted by  $X \tilde{*} Y$  in order to distinguish from the join as the quotient space of  $X \times Y \times \mathbf{I}$ .

The *mapping cylinder* of a map  $f: X \rightarrow Y$  is defined as the space

$$M(f) = X \times [0, 1] \cup Y$$

admitting the topology generated by all open sets in the product space  $X \times [0, 1]$  and the following sets:

$$f^{-1}(V) \times (t, 1) \cup V,$$

where  $V$  is open in  $Y$  and  $0 < t < 1$ . Notice that  $M(f)$  is not a quotient space of  $X \times \mathbf{I} \oplus Y$ . It is easily observed that  $X * Y$  is homeomorphic to

$$M(\text{pr}_X) \cup_{X \times Y \times \{0\}} M(\text{pr}_Y),$$

where  $\text{pr}_X: X \times Y \rightarrow X$  and  $\text{pr}_Y: X \times Y \rightarrow Y$  are the projections. By using the Bing Metrization Theorem, it is easy to see that  $M(f)$  (hence  $X * Y$ ) is metrizable if so are  $X$  and  $Y$ . Extending [Ca<sub>3</sub>, Lemma 6.3], we can show the following:

LEMMA. For any map  $f: X \rightarrow Y$ , the mapping cylinder  $M(f)$  is stratifiable if so are  $X$  and  $Y$ .

By [Hy] (cf. [KL]),  $M(f)$  (hence  $X * Y$ ) is an  $\text{ANR}(\mathcal{M})$  if so are  $X$  and  $Y$ . This is expected to be true for  $\text{ANR}(\mathcal{S})$ 's. However we cannot apply this method to stratifiable spaces (cf. [Ca<sub>1</sub>]). In fact, San-ou [Sa] constructed a stratifiable space  $X$  with  $A$  a closed set such that  $(X, A)$  is not semi-canonical. (For the definition of semi-canonical pairs, refer to [Hy].) In his construction, by replacing  $\mathbf{N}$  and  $\mathbf{Q}$  by  $\mathbf{R}$ , we have a stratifiable locally convex linear topological space  $X$ , hence  $X$  is an  $\text{AR}(\mathcal{S})$ , such that  $(X, A)$  is not semi-canonical, where  $A = \{0\}$ . Consider the mapping cylinder  $M(i)$  of the inclusion  $i: X \setminus A \subset X$ . Then  $(M(i), X)$  is not semi-canonical. And  $((X \setminus A) * X, X)$  is not semi-canonical. Thus we need another approach.

To characterize  $\text{AR}$ 's, Borges [Bo] introduced the concept of hyperconnectedness. For a space  $X$ , let  $F(X)$  be the full simplicial complex with  $X$  the set of vertices, i.e.,  $X = F(X)^{(0)}$ . Introducing a topology on  $|F(X)|$ , Cauty [Ca<sub>4</sub>] constructed a test space  $Z(X)$  such that a stratifiable space  $X$  is an  $\text{ANR}(\mathcal{S})$  if and only if  $X$  is a neighborhood retract of  $Z(X)$ . Improving the construction of  $Z(X)$ , Miwa [Mi] constructed a hyperconnected space  $E(X)$  containing  $X$  as a closed set and proved that  $E(X)$  is stratifiable if so is  $X$ . Then any stratifiable space  $X$  can be embedded in an  $\text{AR}(\mathcal{S})$   $E(X)$  as a closed set. By his construction, any map  $f: X \rightarrow Y$  extends to the map  $\tilde{f}: E(X) \rightarrow E(Y)$  which is a simplicial map from  $F(X)$  to  $F(Y)$ . For this extension  $\tilde{f}$ , we have the following:

THEOREM 1. Let  $\tilde{f}: E(X) \rightarrow E(Y)$  be the extension of a map  $f: X \rightarrow Y$ . Then  $M(\tilde{f})$  is hyperconnected. Hence  $M(\tilde{f})$  is an  $\text{AR}(\mathcal{S})$  in case  $X$  and  $Y$  are stratifiable.

Since  $M(f)$  is a closed subset of  $M(\tilde{f})$ , the following problem reduces to prove that  $M(f)$  is a neighborhood retract of  $M(\tilde{f})$ .

PROBLEM 1. Let  $f: X \rightarrow Y$  be a map between  $\text{ANR}(\mathcal{S})$ 's. Is the mapping cylinder  $M(f)$  an  $\text{ANR}(\mathcal{S})$ ?

Although this has not yet been succeeded, the following holds:

THEOREM 2. Let  $X$  and  $Y$  be  $\text{ANR}(\mathcal{S})$ 's and  $f: X \rightarrow Y$  a Hurewicz fibration. Then the mapping cylinder  $M(f)$  is an  $\text{ANR}(\mathcal{S})$ .

Since the projection  $\text{pr}_X: X \times Y \rightarrow X$  is a Hurewicz fibration, we have the following generalization of [Ca<sub>3</sub>, Corollary 6.2]:

THEOREM 3. If  $X$  and  $Y$  are  $\text{ANR}(\mathcal{S})$ 's then so is the join  $X * Y$ .

*Remark.* We can also prove Theorem 3 by showing that  $E(X) * E(Y)$  is hyperconnected and that  $X * Y$  is a neighborhood retract of  $E(X) * E(Y)$ . This approach is easier than the above approach.

In [Ca<sub>2</sub>], Cauty asserted that the adjunction space of  $\text{ANR}(S)$ 's is also an  $\text{ANR}(S)$ , but his key lemma is false [Sa] (even if  $(X, A)$  is a pair of  $\text{ANR}(S)$ 's as shown in the above). Thus his assertion is still a conjecture and Theorem 3 is still open for the quotient topology:

**PROBLEM 2.** *Let  $X$  and  $Y$  be  $\text{ANR}(S)$ 's. Is the join  $X * Y$  with the quotient topology an  $\text{ANR}(S)$ ? For any map  $f: X \rightarrow Y$ , is the mapping cylinder  $M(f)$  with the quotient topology an  $\text{ANR}(S)$ ?*

In [Ca<sub>3</sub>], Cauty proved that the direct limit of the tower of compact  $\text{ANR}(\mathcal{M})$ 's is an  $\text{ANR}(S)$ . It is natural to ask the following:

**PROBLEM 3.** *Let  $X_1 \subset X_2 \subset \cdots$  be a tower of  $\text{ANR}(S)$ 's such that each  $X_{n+1}$  is a closed subspace of  $X_n$ . Is the direct limit  $\text{dir lim } X_n$  an  $\text{ANR}(S)$ ?*

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